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Reparametrizing the Antecedent Moisture Model

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Abstract

Developing simple and accurate hydrologic models for wet-weather sanitary sewer flow has long been enigmatic for the engineering community. Hydrologic models suitable for surface water hydrology often perform poorly when simulating sanitary sewer hydrology when they do not account for antecedent moisture conditions and seasonality. The result is underpredicting peak flow and volume in wet conditions and overpredicting the same in dry conditions.

The Antecedent Moisture Model is an empirically-calibrated method which models a rainfall capture fraction that varies by antecedent moisture and season and transforms the captured rainfall into a flow hydrograph. The Antecedent Moisture Model has been particularly successful in modeling sanitary sewer infiltration, for which other methods perform poorly.

This paper suggests a reparameterization of the original model which provides virtually identical results while improving interpretability and ease of use. The reparametrized model is designed to be timestep independent and more physically relatable, intuitive, and scaleable than the original parameterization while remaining functionally equivalent to the original parameterization. The paper also presents two computational examples which demonstrate the application of the Antecedent Moisture Model formulas and math.

1 Introduction

1.1 Background

Rainfall-runoff-sewage management systems are prevalent throughout the built environment, including stormwater systems, combined sewer systems, separate sanitary sewer systems, and "improved" river systems. Engineers frequently perform modeling of these systems for applications like flood control, combined sewer overflow (CSO) control, and sanitary sewer overflow (SSO) control. Hydrologic models that accurately relate flow to rainfall are critical for designing rainfall-runoff-sewage management systems which may cost communities many millions of dollars.

Modeling of sanitary sewer rain-derived infiltration and inflow (RDII) has been especially enigmatic because of its multiple and complicated pathways (Water Environment Federation, 2017). Wet-weather flows enter the sewer through a variety of pathways including leaky manholes, mains, and laterals. RDII can also come from sump pumps and directly-connected foundation drains which are no longer allowed in new construction, but were allowed in the past. Many of these pathways pass through the soil and resulting RDII is affected by antecedent (pre-event) soil moisture condition.

Studies of streamflow have shown the importance of soil moisture on flow generation for some pathways and RDII is analogously affected by soil moisture—positive pore-water pressure is required for subterranean water to infiltrate pipes or manholes and this changes seasonally (Pangle et al., 2022).

Even though separate sanitary sewer systems nominally contain no direct clear water connections, the authors have observed catchments in the Midwest in which peak wet-weather flows can reach 5 to 50 times dry-weather flows.

The United States Environmental Protection Agency (US EPA) promotes the use of the RTK method for estimating wet weather flow as part of its Sanitary Sewer Overflow Analysis and Planning (SSOAP) Toolbox (US EPA, n.d.). RTK was selected as the preferred method during a review of available methods to estimate wet weather flow in sanitary sewers (US EPA 2008). The RTK method has been incorporated into the widely used SWMM model. The R parameter in RTK denotes capture fraction (alternately known as percent capture), which is the proportion of rain falling over the catchment that is converted into flow. The RTK method assumes a constant capture fraction, regardless of antecedent moisture conditions and depth of storm. This is a weakness of the RTK method since many systems respond with RDII that increases with antecedent moisture conditions (Czachorski, 2001). Seasonal differences can be incorporated into the RTK method by changing RTK parameters monthly, which increases the number of calibration parameters by up to a factor of twelve.

Czachorski (2022) presented the Antecedent Moisture Modeling (AMM) method which was developed to account for antecedent moisture and seasonal effects in predicting rainfall runoff. The equations have been peer reviewed and tested on hundreds of systems over the past 25-years ranging from separate sewers to combined sewers to surface water runoff. The equations have been placed in the public domain (Czachorski 2021a) and guides have been published on how to use the model for design (Czachorski (2021b). The equations have been demonstrated to accurately simulate antecedent moisture (AM) dynamics in sanitary sewer systems (Czachorski et al. 2001; Van Pelt et al. 2002; Czachorski et al. 2005; Czachorski et al. 2006; Kuhns et al. 2007; Ricker et al. 2008; Kuhns et al. 2010; Czachorski et al. 2011; Kuehlmann et al. 2013; Czachorski et al. 2014).

1.2 Enhancements to the Original Equation

The original AMM equations documented by Czachorski (2022) were derived using the principles of system identification from the field of aerospace control systems to find the simplest mathematical model with the fewest parameters that accurately describes the relationship between system inputs and the flow output. There are four types of enhancements to the original form of the equations addressed in this paper:

1. Improving physical relatability – Reparameterized the original model so that the parameters are intuitively understandable to users and the units are explicitly stated.
2. Creating scale independence – With the original equations, scaling a calibrated AMM model to a catchment of a different size requires scaling 3 different parameters. The new parameterization explicity includes catchment area which allows direct comparisons in parameter values between differently-sized catchments.
3. Creating timestep independence – With the original equations, some parameters were timestep dependent, requiring conversion to different timestep. The new equations have parameters that are independent of timestep.

4. Updating nomenclature – Some minor improvements were made to make the nomenclature simpler and more intuitive.

1.3 Purpose

This paper presents refinements to the AMM equations that address the issues outlined above. The functionality of the model remains unchanged, but the authors believe the reparametrized equations yield a more intuitive, orthogonal, and user-friendly implementation of the AMM model as well as adding the benefit of timestep independence.

2 Model Reparameterization

2.1 Overview of Changes

The original equations have been reparameterized to enhance the equations as described above. The revised equations are consistent with the functionality of the original implementation presented by Czachorski in 2022. A model calibrated to the original equations can be transformed to the reparametrized model with near-identical results.

Enhancing the equations as described makes them appear more complex in their parameterization. However, the authors believe that this reparameterization is worthwhile to enhance understanding and usability for the user in applications. Importantly, although the reparameterization changes the appearance of the equations, as noted earlier, the new equations are mathematically equivalent to the old equations, just expressed in a more convenient form, with two minor exceptions that will be discussed in Section 3.

Improving Physical Relatability

In the new parameterization units have been added to all equations. Properly notating equations with units helps ensure consistent implementation and allows translation between different unit systems.

The new parameterization of the equations explicitly refers to rainfall capture fraction, or the fraction of rainfall falling over the catchment which is eventually captured as flow. The capture fraction (which increases with increasing wetness) can be easily plotted over time. This is an intuitive metric which permits better understanding of model significance and of scaling to other catchments.

Time to peak is easily incorporated through rainfall averaging. This allows modeling of the time delay between the flow response and the rain.

The new parameterization of the equations is also analogous to the RTK method. This makes the AMM model easier to conceptualize for those already familiar with RTK. The similarities to AMM will be covered in more detail later in this paper.

Creating Scale Independence

A catchment area parameter has been added to the new equations and the calibratable parameters have been changed to refer to rainfall capture fraction. This means calibratable parameters between two otherwise identical catchments that differ only in size will be the same and only the area will vary. This allows for direct comparison of parameters between catchments.

The rainfall capture fraction normalization was chosen for its consistency with the RTK method and intuitiveness. Other normalization techniques, such as flow per linear feet of pipe per inch of rain, could be used in a similar way.

Creating Timestep Independence

With timestep independence, results calculated with two different timesteps will yield similar results. This allows for comparisons of model parameters across systems and models, regardless of the timestep.

Two features are required for timestep independence:

- decay parameters in terms of half-lives that are not dependent on the model timestep, and
- minimizing approximation errors from numerical estimation of the system of differential equations.

Updating Nomenclature

Finally, nomenclature has been updated for clarity. The "*High T*" and "*Low T*" parameters have been renamed "*Cold Temp*" and "*Hot Temp*" to prevent confusion. After reparameterization the original "Temperature Factor" used different units so it was clear it needed a new name to prevent confusion. It was renamed the "Seasonal Hydrologic Condition Factor" (*SHCF*).

Some new variables, such as for the timestep and decay half-lives, have been added and named intuitively.

2.2 New Parameterization Equations – Standard AMM Component

The first two levels of the AMM method are the rainfall runoff function and the antecedent moisture function.

The equations are refactored in terms of capture fraction which the authors find more intuitive than the Response Factor (*RF*) term used in Czachorski (2022). *RD* is the capture fraction during very dry conditions and is a constant parameter. *RW* is the additional capture fraction due to antecedent moisture conditions and increases or decreases as wetness conditions change. The total capture fraction of a unit of rainfall at any point in time is equal to *RD* + *RW*.

A moving average is applied to the precipitation in Equation 1 to allow the AM model to respond fully to a one time step precipitation increment over several time steps in the manner of a typical unit hydrograph.

The original equations used in Czachorski (2022) were timestep-dependent. Consequently, models could not be used at timesteps other than the one for which they were calibrated without refactoring the decay parameters. The reformulated equations are timestep-independent so that parameter values and results may be freely compared between models at different timesteps.

Level 1 – Rainfall Runoff Function

The reparametrized Level 1 equations are as follows:

$$Q_t = A * \left(RD + \frac{RW_t + RW_{t-1}}{2}\right) * MAP_t * \frac{(1-SF)}{\Delta t} + SF * Q_{t-1} \quad (1)$$

$$SF = 0.5^{(\Delta t/HHL)} \quad (2)$$

$$MAP_t = \frac{1}{\frac{PAT}{\Delta t}+1} \sum_{i=1}^{\frac{PAT}{\Delta t}+1} P_{t-i} \quad (3)$$

where:

Q_t = flow rate at the current time step t [L^3/T],
A = catchment area [L^2],
RD = minimum rainfall capture fraction during dry weather [·],
RW_t = additional rainfall capture fraction during wet weather [·],
SF = constant shape factor of hydrograph, bounded by [0, 1] [·],
Δt = model time step [T],
Q_{t-1} = flow rate at the previous time step $t-1$ [L^3/T],
MAP_t = moving average incremental precipitation at time step t, [L],
PAT = precipitation averaging time of the catchment (integer multiple of Δt) [T],
P_{t-i} = incremental precipitation starting at time step $t-i$, where i varies from 1 to $PAT/\Delta t+1$ [L], and
HHL = hydrograph half-life [T].

PAT should be an integer multiple of Δt. Additional modifications can be made to handle non-integer increments, and have been made in the "AMM-for-PCSWMM" implementation referenced at the end of this paper.

Incremental rainfall timeseries can be "start-of-interval", meaning that each rainfall depth is assumed to occur at the start of its associated date/time value and last for a time equal to the gauge's recording interval, or they can be "end-of-interval", meaning the time stamp associated with a rainfall value is for the end of the recording interval (US EPA 2016). Equation 3 assumes the "start-of-interval" convention, though it can easily be adapted to use the "end-of-interval" convention.

Although Equation 3 looks complex, it is a rather simple moving average equation that for $PAT > 0$ has the effect of spreading a unit of rainfall out over several time steps. In the case that the time to peak $PAT = 0$, indicating that the system fully responds by the end of the time step, Equation 3 is reduced to $MAP_t = P_t$.

The time to peak of the unit hydrograph is given by:

$$TP = PAT + \Delta t \tag{4}$$

where:

TP = time to peak of the unit hydrograph [T].

The time to peak is measured from the beginning of rainfall to the peak of flow whereas *PAT* represents the time from the end of the rainfall to the peak of flow. For time scale independence *PAT* must be kept constant if the model timestep is changed. With Equations 3 and 4 the model can be calibrated using either *PAT* or *TP*, but the *TP* parameter is timestep-dependent and should be translated to the equivalent *PAT* parameter prior to converting timesteps.

The $(1-SF)/\Delta t$ term in Equation 1 is required to match the flow volume with the rainfall capture volume. With all other parameters constant, a site with a long HHL will have a lower peak flow rate than a site with a short HHL, in order that the volume at both sites is equal. (This quirk is shared in common with the RTK Unit Hydrograph.) The integral (volume under the curve) of the function $Q_t=SF*Q_{t-1}$ from $t=0$ to ∞ is $Q_0 *\Delta t/(1-SF)$. Applying $(1-SF)/\Delta t$, as a term in Equation 1 ensures that the model rainfall capture fraction is equal to $(RD+RW)$.

Averaging RW_t and RW_{t-1} over the time step reduces the approximation error caused by discretization of a continuous function.

Level 2 – Antecedent Moisture Function

The revised Level 2 equations are as follows:

$$RW_t = \frac{(AMRF-1)}{ln\,(AMRF)} * \ SHCF_t * MAP_t + AMRF * RW_{t-1} \tag{5}$$

$$AMRF = \ 0.5^{(\Delta t/AMHL)} \tag{6}$$

where:

RW_{t-1}	=	additional wet weather rainfall capture fraction at the previous time step *t−1* [·],
$SHCF_t$	=	seasonal hydrologic condition factor at current time step t [1/L],
AMRF	=	constant antecedent moisture retention factor, bound by [0, 1] [·], and
AMHL	=	antecedent moisture half-life [T].

The output from Equation 5 is a time series of how the additional rainfall capture fraction during wet weather RW_t varies in time. It is similar in form to the hydrograph output from Equation 1, but the additional rainfall capture fraction time series represents how antecedent moisture conditions vary over time and affect the hydrograph response to rainfall. It is added to *RD* to represent the total rainfall capture fraction at any time step.

The same averaged time series MAP_t is used in Equations 1 and 5. This ensures that increases in *PAT* do not increase the total volume and that the rainfall capture fraction represented by *(RD+RW)* is an accurate value.

The *(AMRF-1)/ln(AMRF)* term in Equation 5 functions as a timestep correction factor that ensures identical results for any timestep.

Level 3 – Seasonal Hydrologic Condition Factor

The third level equation remains largely unchanged from the original equations published in Czachorski (2022). However, the units have changed. To prevent confusion the Temperature Factor (*TF*) variable is renamed to Seasonal Hydrologic Conditions Factor (*SHCF*). The new name also better reflects the purpose of the variable.

$$SHCF_t = \left[\frac{L}{1+e^{(-k\{MATemp_t - x_o\})}}\right] + Cold\ SHCF - \frac{11}{12}L \tag{7}$$

$$L = 1.2 \times (Cold\ SHCF - Hot\ SHCF) \tag{8}$$

$$k = \left[\frac{4.7964}{(Cold\ Temp - Hot\ Temp)}\right] \tag{9}$$

$$x_0 = \frac{(Cold\ Temp + Hot\ Temp)}{2} \tag{10}$$

$$MATemp_t = \ \frac{1}{\frac{TAT}{\Delta t}+1}\sum_{i=1}^{\frac{TAT}{\Delta t}+1} Temp_{t-i} \tag{11}$$

where:

Cold SHCF = cold seasonal hydrologic condition factor value from Point 1 [1/L],
Hot SHCF = hot seasonal hydrologic condition factor value from Point 2 [1/L],
Cold Temp = cold temperature value from Point 1 [Temp],
Hot Temp = hot temperature value from Point 2 [Temp],
$MATemp_t$ = moving average temperature at time step t [Temp],
TAT = temperature averaging time of the catchment (increments of Δt) [T],
$Temp_{t-i}$ = air temp at time step $t-i$, where i varies from 0 to $TAT/\Delta t$ [Temp],
Point 1 = location in sigmoid function that represents 11/12 of maximum range of seasonal hydrologic condition factor (see Figure 1),
Point 2 = location in sigmoid function that represents 1/12 of maximum range of seasonal hydrologic condition factor (see Figure 1),
L = range of variation of the seasonal hydrologic condition factor [1/L],
x_0 = midpoint between the hot and cold temperature values [Temp], and
k = a factor indicating the sigmoid function steepness [1/Temp].

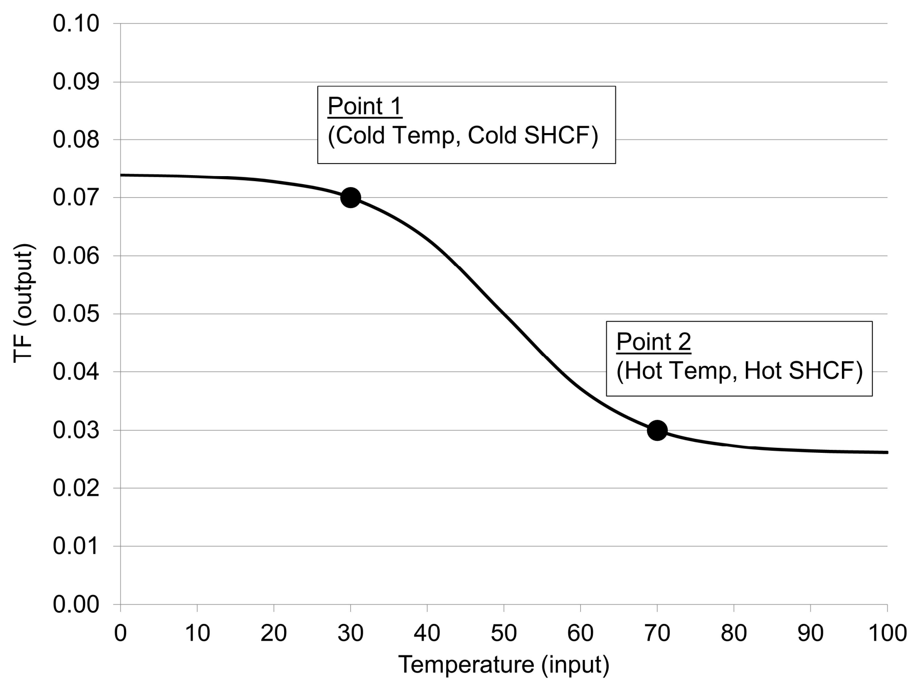

Figure 1 Plot showing the temperature versus the seasonal hydrologic condition factor with the sigmoid function defined by two points on the function; TF Sigmoid: (30, 5) and (70, 1).

The seasonal hydrologic condition factor $SHCF_t$ is the driving time series for the additional wet weather rainfall capture fraction RW_t in Equation 4. When $SHCF_t$ is high the capture fraction increases quickly in response to additional rainfall. When $SHCF_t$ is low more rainfall is needed to increase RW_t by a similar amount. In the Midwest $SHCF_t$ is typically highest in late winter and early spring when vegetation is not active and evapotranspiration is low and lowest in late summer when vegetation is active and evapotranspiration is high.

In practice, using a different *Cold SHCF* parameter in the first and second halves of the year can sometimes be used to produce better results. This is somewhat of a workaround existing AMM users have found to improve model performance in the Upper Midwest. Future research could work to identify a more unified methodology.

Calibratable Parameters

While there are many variables used in the above equations, many of them are either calculated or are fixed during calibration. Only 6 parameters are typically calibrated, and these can be roughly thought of as follows:

1. *RD* – Rainfall Capture Fraction during very dry conditions
2. *PAT* – Precipitation averaging time
3. *HHL* – Hydrograph recession time
4. *AMHL* – Drying time; duration of antecedent moisture effect
5. *Cold SHCF* – Unit increase in Rainfall Capture Fraction in winter
6. *Hot SHCF* – Unit increase in Rainfall Capture Fraction in summer

In practice AMHL is difficult to calibrate if the observed data does not contain a series of back-to-back storms and at times may need to be estimated based upon user experience. An additional *Cold SHCF* parameter may be added to represent conditions in the Fall.

A simplified model could be comprised of only four parameters (*HHL*, *AMHL*, *Cold SHCF* and *Hot SHCF*). These four parameters alone are sometimes sufficient to develop an adequate model that accounts for antecedent moisture effects when both *RD* and *PAT* are set to zero. *RD* and *PAT* are included in the equations because they are useful additions to the model for simulating some systems. The simplification of the model to four core parameters is noted here to highlight the simplicity and parsimony of the model, which makes it easier to use and understand and provides more confidence in the ability to make predictions.

2.3 New Parameterization Equations – Base Flow Component

Equivalent transformations can be made for the baseflow component, which omits Level 2 and uses only Levels 1 and 3. In this formulation the capture fraction varies as a direct function of air temperature, without reference to precipitation. Eliminating the Level 2 equation for wetness conditions from preceding rainfall works well for simulating base flow because the base flow has a very long response time, with precipitation averaging time *PAT* values often in the range of 10 to 20 days in the Midwest. These long *PAT* values capture the impacts of preceding rainfall on the base flow without the need for the Level 2 equation, thus making the base flow model simpler and with fewer parameters. The *RD* term of Equation 1 is not required and is dropped in Equation 12 and RW_t is changed to R_t to reflect that it's the total (not additional) capture fraction. Equations 2 and 3 are used in identical form.

$$Q_t = A * \left(\frac{R_t + R_{t-1}}{2}\right) * MAP_t * \frac{(1-SF)}{\Delta t} + SF * Q_{t-1} \qquad (12)$$

$$R_t = \left[\frac{L}{1+e^{(-k\{MATemp_t - x_o\})}}\right] + Cold\ R - \frac{11}{12}L \qquad (13)$$

$$L = 1.2 \times (Cold\ R - Hot\ R) \qquad (14)$$

$$k = \left[\frac{4.7964}{(Cold\ Temp - Hot\ Temp)}\right] \tag{15}$$

$$x_0 = \frac{(Cold\ Temp + Hot\ Temp)}{2} \tag{16}$$

where:
Cold R = cold rainfall capture fraction from Point 1 [·] and
Hot R = hot rainfall capture fraction from Point 2 [·].

Calibratable Parameters

For the base flow component, only 5 parameters are typically calibrated, and these can be roughly thought of as follows:

1. PAT – Precipitation averaging time
2. HHL – Hydrograph recession time
3. Cold R – Rainfall Capture Fraction in winter
4. Hot R – Rainfall Capture Fraction in summer

2.4 Model Structure and Interpretation

Physical Meaning of Model Parameters

One of the key advantages of the AMM equations reparameterization presented in this paper is that the model parameters and predictions have physical meaning. The intuitive physical meaning of all six input parameters leads to a deeper understanding of the how the model works and helps identify the parameters during calibration.

The model predicts three output time series from equations 1, 5 and 7. These output predictions have the following physical meaning:

- Level 1 – Flow Q is the prediction of flow from the catchment.
- Level 2 – Wet Weather Capture Fraction *RW* is the prediction of how the wet weather rainfall capture fraction changes in time due to antecedent moisture conditions. Predicting how the capture fraction changes in time is the holy grail of wet weather flow modeling.
- Level 3 – Seasonal Hydrologic Condition Factor *SHCF* is the prediction of the rate of increase in the wet weather capture fraction from rainfall, with higher values in spring when it is cold and lower values in summer when it is hot.

The physical meaning of the model parameters and output, combined with the parsimonious equations and parameters results in a powerful model capable of accurately simulating rainfall runoff dynamics for a variety of systems under varying antecedent moisture conditions.

Parallels to the RTK Hydrologic Model

The reparametrized equations intentionally clarify parallels to the RTK method. Figure 2 shows the response of both the reparameterized equations and RTK method to 1 unit of rainfall between time 2 and 3 where the time to peak is 3. The total rainfall capture fraction of ($RD + RW_t$) is equivalent to the "R" in the RTK model (but is modeled as variable over time), the time to peak $TP = PAT +$

Δt for the moving average precipitation is equivalent to the "T" in the RTK model, and the hydrograph half-life *HHL* is analogous to *K*T* in the RTK model.

The RTK method is a special case of the AMM method for which the *SHCF* parameters are zero (no antecedent moisture effect), only AMM uses a smooth exponential rise and decay rather than a triangular hydrograph.

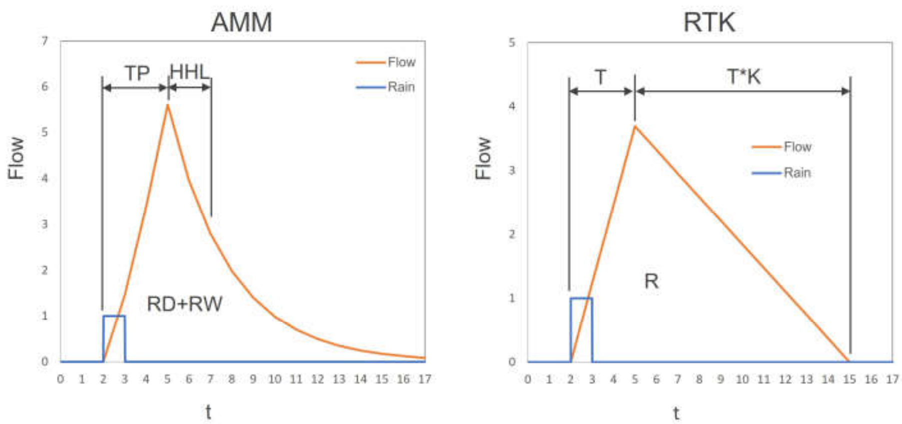

Figure 2 Visual comparison of RTK and AMM responses to a unit rainfall over one timestep.

Model Component Recommendations

A complete antecedent moisture model of a sewer system in the Midwest typically requires multiple AMM components acting on different time scales. Czachorski (2022) and this paper describe two different types of AMM components:

- The standard 3-level component represented with Equations 1 to 11 that works well for representing wet weather flow consisting of runoff or rain-derived infiltration or inflow in sanitary sewers. A schematic of this model component is shown in Figure 3.
- A base flow 2-level component represented in Equations 1 to 3 and 12 to 15 that works well for representing base flow or ground water infiltration. A schematic of this model component is shown in Figure 4.

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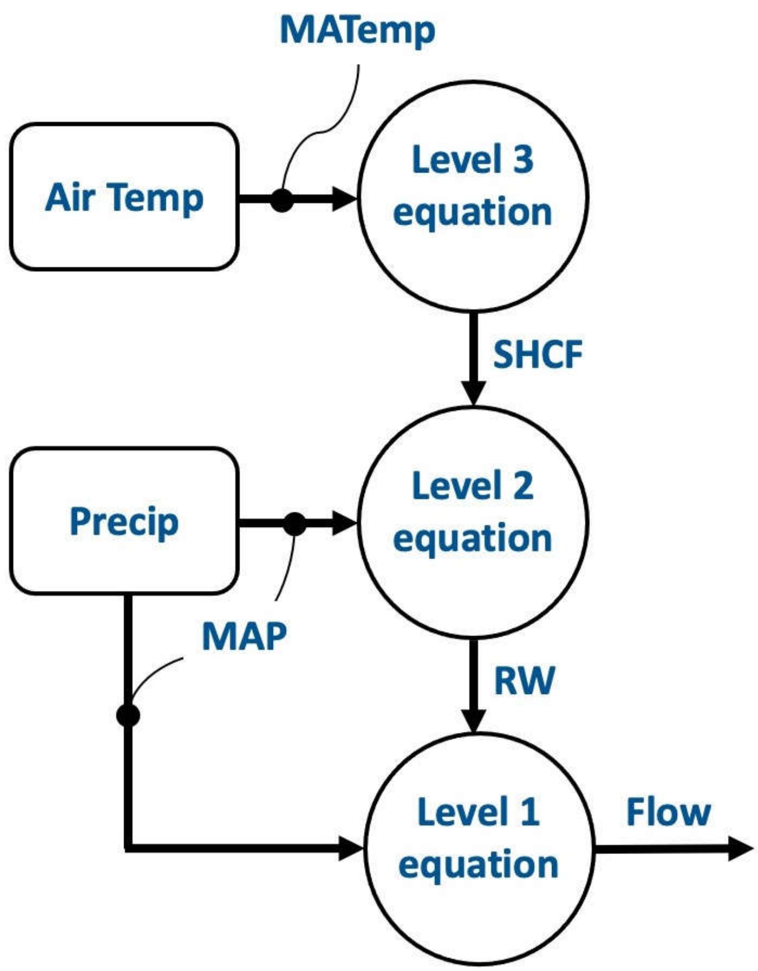

Figure 3 Antecedent moisture model schematic for runoff or rainfall-derived inflow or infiltration in sanitary sewers.

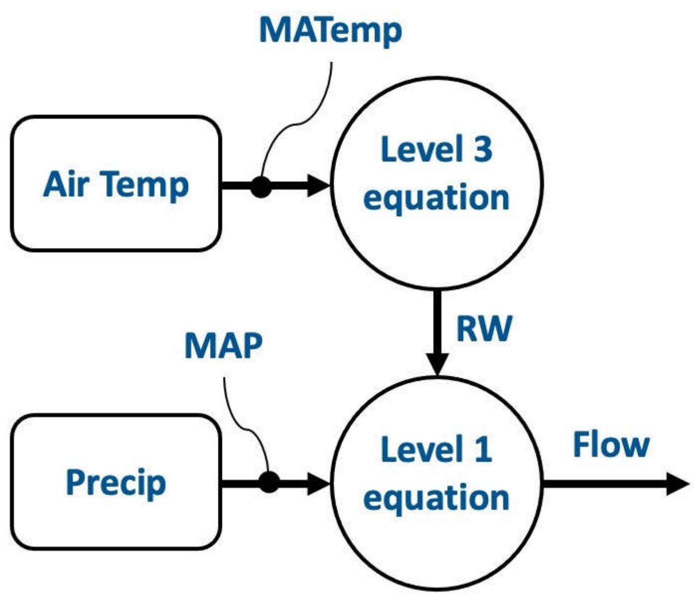

Figure 4 Antecedent moisture model schematic for baseflow or ground water infiltration in sewer systems.

These components can be used in combination to model all the flow in a sanitary sewer system except for the diurnal variation. A typical application would include two 3-level components (Fast and Slow response for inflow and infiltration, respectively) and a single 2-level baseflow component. These flow components can be modeled separately and added together to derive the total system flow as shown in Figure 5.

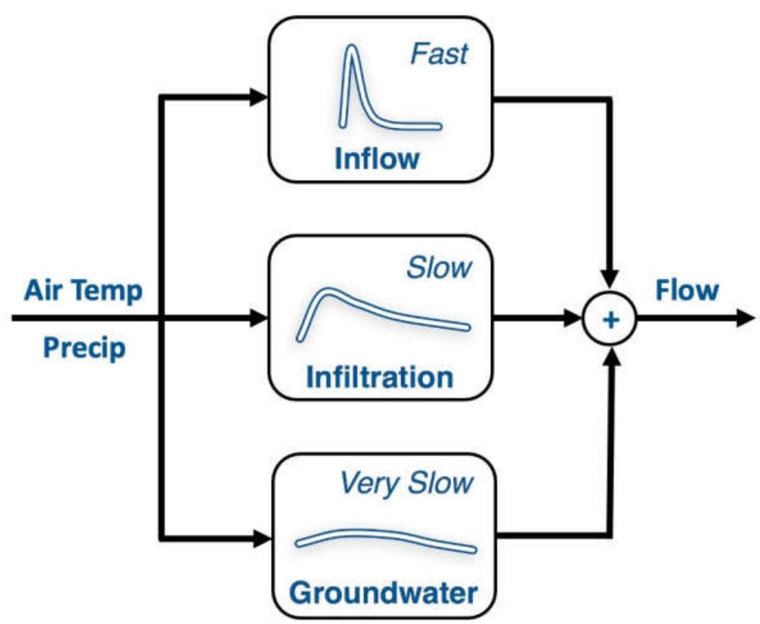

Figure 5 Schematic representation of the antecedent moisture model for sanitary flow components present in sanitary collection systems.

In general, each component should be permitted to use distinct parameters, as it is common for components with longer hydrograph half-lives, like sanitary sewer infiltration, to display a greater degree of antecedent moisture dependence than fast components like runoff or inflow that might be dominated by impervious areas not as sensitive to antecedent moisture.

The modeler or analyst should use observations of system data and judgment to determine the number of components required to accurately represent a catchment.

2.5 Approximation Errors

Reformulating parameters in terms of half-lives makes it easy to recalculate results at a different timestep. But without properly accounting for differing systematic approximation errors using different timesteps the results could still vary.

Approximation error here is meant to refer to a discrepancy between the model flows for some value of Δt being modeled and the "exact" model flow (as the timestep Δt approaches zero).

A second type of approximation error is introduced when the rain time series increment is not much shorter than the Hydrograph Half Life *HHL*. This type of approximation error is problematic for any rainfall-runoff method, not just AMM. This is not the type of approximation error currently being considered. It is the authors' intention to minimize the first type of approximation error (which can be dealt with in the equation setup) so that users can generally consider the first type insignificant compared to the second type.

Equations 1 and 5 are approximate solutions to the following set of differential equations governing the assumptions of the AMM method:

$$\frac{dQ}{dt} = \frac{A*(1-SF)}{\Delta t} * (RD + RW) * I - \frac{\ln 2}{HHL} * Q \tag{17}$$

$$\frac{dRW}{dt} = RD * SHCF * I - \frac{\ln 2}{AMHL} * RW \tag{18}$$

where:

Q = flow timeseries [L^3/t],
RW = additional rainfall capture fraction timeseries [·], and
I = rain intensity timeseries [L/t].

Equation 17 is simple enough to solve exactly. Equation 5 is its exact solution. The second term of Equation 16 is similarly simple to solve and the second term of Equation 1 is an exact solution.

The first term of Equation 16 is not trivial to solve as it represents a system of two first-order ODEs. The original equations in (Czachorski, 2022) use a single value of the Level 2 function (*RF*) for approximation of the new flow value, which can result in a poor approximation when the value of the Level 2 function changes significantly during a timestep. The first term of Equation 1 in this paper uses an average of the Level 2 function (*RW*) at the beginning and end of the timestep, significantly improving approximation accuracy.

The original equations in (Czachorski, 2022) represent what is nearly an Euler method approximation. The first term of Equation 1 in this paper is a non-standard two-stage, first-order Runge-Kutta approximation that is still relatively simple but more accurate. Better approximations exist, but the authors consider Equation 1 to be a good compromise between accuracy and minimal complexity that will not hinder adoption.

The approximation error for the reparametrized equations is always biased negative: longer timesteps will always decrease flow estimates compared to shorter timesteps. In one representative but stressing test case the maximum flow approximation error for a 1-hour timestep was found to be -33.8% for the original equations and -1.5% using the reparametrized equations.

It is generally more accurate to use a shorter (5- or 15-minute) timestep and rain data increment in calibration, when data is available, to minimize approximation error. Changes to the timestep or rain data increment after calibration should only be made with care to understand the effects. Approximation errors of the first type mentioned in this section are expected to be smaller than of the second type in most cases.

3 Parameter Transformations

It is useful to provide parameter transformations to translate parameters between the original AMM model in (Czachorski, 2022) and the reparametrized model. Use of these transformations will allow conversion of calibrated model parameters from the original model equations to the new parameterization equations.

Only two changes were made which are not reconcilable to the original equations.

First, Level 2 of the original equations applies the precipitation of the immediately prior timestep P_{t-1}, rather than that of the current timestep P_t, as shown in Equation 5 in (Czachorski, 2022). This was intended to prevent precipitation from instantaneously modifying its own moisture condition, which can result in problematic overpredictions for course timesteps. However, applying different precipitation values to Level 1 and Level 2 is in conflict with creating timestep independence. Rather, the preferred method to avoid the issue is to use a shorter timestep to reduce approximation error. To create timestep independence the precipitation term in Equation 5 of the current paper has been made identical to that in Equation 1 (essentially changing the original P_{t-1} term to P_t).

Second, the use of the $(RW_t+RW_{t-1})/2$ term in Equation 1 significantly reduces approximation error compared to using a single timestep RW_t, but also results in irreconcilable differences compared to the original model.

Were it not for these small changes, any model calibrated with the translated equations could be translated exactly to a reparametrized model. With this change there will be some small difference in a translated model. For the example using an hourly timestep in the companion spreadsheet to this paper (see the Resources section later), the maximum difference at any timestep was 5.8%.

Additionally, due to explicit treatment of start-of-interval vs. end-of-interval rainfall in this paper the flow generated by the reparametrized model will be delayed by one timestep compared to flow generated by the original equations.

The equations below show the conversion from the original equation parameters (Czachorski, 2022) to the new equations parameters.

$$RD = \frac{-AC*\Delta t}{A*(1-SF)} \qquad (19)$$

$$HHL = \frac{-ln(2)*\Delta t}{ln(SF)} \qquad (20)$$

$$AMHL = \frac{-ln(2)*\Delta t}{ln(AMRF)} \qquad (21)$$

$$PAT = \frac{Original\ timesteps\ averaged\ for\ precip}{\Delta t} - 1 \tag{22}$$

$$SAT = \frac{Original\ timesteps\ averaged\ for\ temp}{\Delta t} - 1 \tag{23}$$

$$Cold\ Temp = High\ T \tag{24}$$

$$Hot\ Temp = Low\ T \tag{25}$$

$$Cold\ SHCF = High\ TF * \frac{ln(AMRF)*\Delta t}{(AMRF-1)*A*(1-SF)} \tag{26}$$

$$Hot\ SHCF = Low\ TF \frac{ln(AMRF)*\Delta t}{(AMRF-1)*A*(1-SF)} \tag{27}$$

4 Model Applications

Two example model computations are provided below. All example computations are available in the in the companion spreadsheet to this paper (see the Resources section later) within tabs 20 and 23.

4.1 Computational Example

The computations for this model are shown below and can be accessed in Tab 23 of the companion spreadsheet. This example illustrates the relative simplicity of the model computations recursively in a spreadsheet format.

For this example, a rainfall input of 1 in. (25.4 mm) is assumed for four subsequent time steps, with a slightly varying air temperature to show the impact of the Season Hydrologic Condition Factor. The example computations are shown for the following input parameters:

Catchment Area (*A*)	=	1000 ac (404 ha)
Hydrograph Half Life (*HHL*)	=	2 h
Dry-weather Capture Fraction (*RD*)	=	0.01 unitless
Antecedent Moisture Half Life (*AMHL*)	=	8 h

	Temp [°F]	SHCF [1/in.]
Cold	30	0.07
Hot	70	0.03

The shape factor SF and antecedent moisture retention factor *AMRF* are computed from Equations 2, and 5, respectively, as shown below. Note that the model timestep Δt = 1 h.

$SF = 0.5^{(1/2)} = 0.707$

$AMRF = 0.5^{(1/8)} = 0.917$

The precipitation averaging time *PAT* was set to zero, eliminating Equation 3 and resulting in the precipitation time series as the direct input into Equation 1. This *PAT=0* assumption has been observed by the authors to be quite satisfactory when modeling small catchments. The moving average temperature computations are not shown for this example and the *MAT* time series is shown as an input time series for simplicity.

The Seasonal Hydrologic Condition Factor $SHCF_t$ is computed using equations 7 – 10 from the Sigmoid Function. Example computations of the Sigmoid Function Parameters and the $SHCF_t$ for the first timestep are shown below.

L = 1.2*(0.070-0.030) = 0.048

K = 4.7964*(30-70) = -0.1199

x_0 = (30+70)/2 = 50

$SHCF_t$ = (0.048/(1+e^(-(-0.1199)*(70-50))))+0.07-11/12*0.048 = 0.0300

Figure 6 shows a plot of the temperature versus the seasonal hydrologic condition factor resulting from these computations. Note how the sigmoid function is close to linear for moderate temperatures but limits the high and low range of the *SHCF* values.

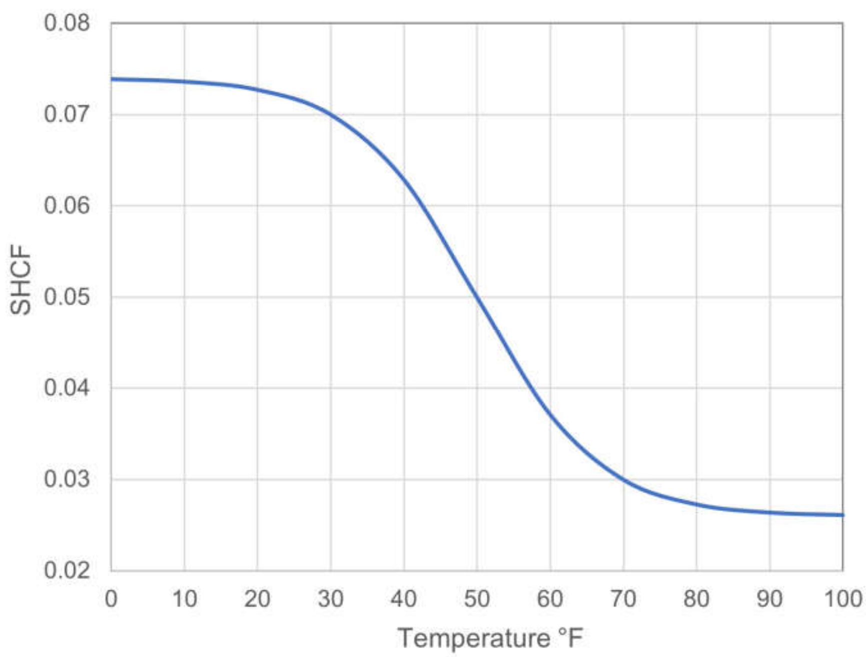

Figure 6 Plot showing with Temperature versus the Seasonal Hydrologic Conditions Factor using the Sigmoid Function.

The results of the model output for these inputs and assumptions are shown in Table 1.

Table 1 Example computations of the reparametrized model.

Time	Precip.	Moving Average Precip.	Moving Average Temp.	Seasonal Hydrologic Condition Factor	Additional Capture Fraction	Flow Output

T	P_t	MAP_t	$MATemp_t$	$SHCF_t$	RW_t	Q_t
h	in.	in.	°F	1/in.	unitless	cfs
Time	Input	Level 1 Eq. 3	Input	Level 3 Eq. 6	Level 2 Eq. 4	Level 1 Eq. 1
0:00	0	0	70.1	0.0300	0.0%	0.00
1:00	1	0	70.0	0.0300	0.0%	0.00
2:00	1	1	69.9	0.0300	2.9%	7.20
3:00	1	1	69.8	0.0301	5.3%	20.05
4:00	1	1	69.7	0.0301	7.7%	36.24
5:00	0	1	69.6	0.0302	10.6%	55.60
6:00	0	0	69.5	0.0302	9.7%	39.32
7:00	0	0	69.4	0.0303	8.9%	27.80
8:00	0	0	69.3	0.0303	8.2%	19.66
9:00	0	0	69.2	0.0304	7.5%	13.90
10:00	0	0	69.1	0.0304	6.9%	9.83

Note the decay of both the flow rate *Q* and the additional capture fraction *RW* in Table 1. The flow rate in each time step after the precipitation ends at 5:00 is 70.7% of the flow rate in the previous timestep because of the Shape Factor *SF* decay of 0.707. Similarly, the additional capture fraction *RW* in each time step after 5:00 is 91.7% of the flow in the previous timestep because of the Antecedent Moisture Retention Factor *AMRF* decay of 0.917.

Figure 7 shows a plot of the additional capture fraction *RW* and the flow rate *Q*. Note how the additional capture fraction increases during the rain and recedes after the rainfall has ended. This shows graphically how the capture fraction varies during and after storms, indicating the level of antecedent moisture conditions in the catchment.

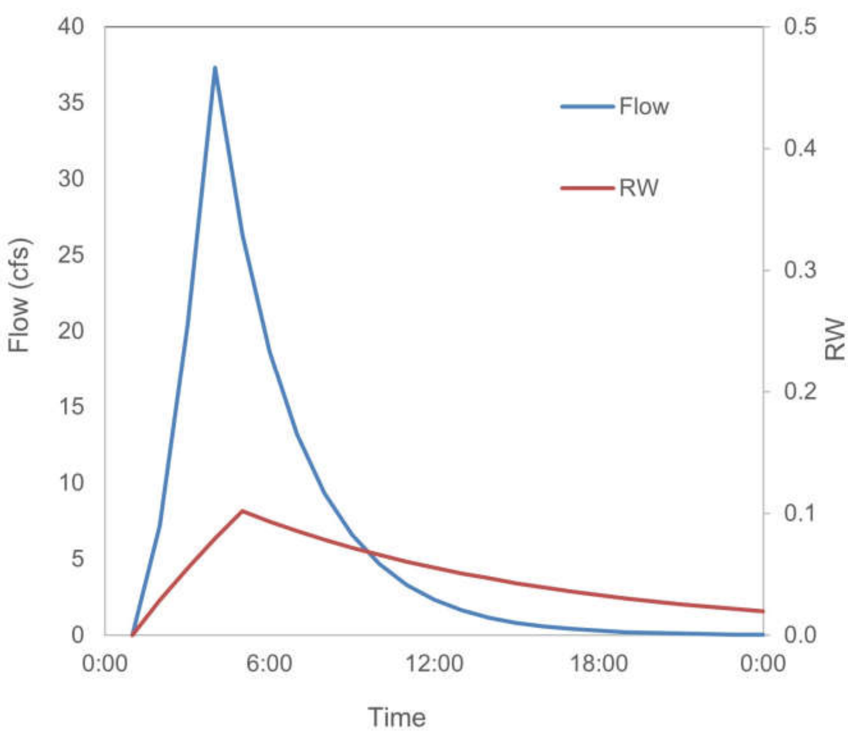

Figure 7 Plot showing the model out for flow and additional wet weather capture fraction.

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Table 2 shows the formulas for several rows of the model to illustrate the recursive computations for several timesteps, which are easily reproduced in a spreadsheet. The formulas reference the row numbers and column letters added to the top left side of the table. The table shows the three levels of the model as follows:

- Level 1 is the output flow rate Q_t shown in Column G that is a function of the rainfall input and the total rainfall capture fraction computed from the Dry-weather Capture Fraction *RD* and the Additional Capture Fraction from Wet Weather RW_t,
- Level 2 is the Additional Capture Fraction from Wet Weather RW_t shown in Column F that varies in response to antecedent moisture conditions as a function of rainfall and air temperature for seasonal effects.
- Level 3 is the Season Hydrologic Condition Factor $SHCF_t$ shown in Column E that varies seasonally as a function of temperature and indicates how much the Additional Rainfall Capture Fraction increases for each inch of rainfall.

Table 2 Formula computations of the reparametrized model.

	A	B	C	D	E	F	G
	t	P_t	MAP_t	MA-Temp$_t$	$SHCF_t$	RW_t	Q_t
			Level 1 Eq. 3		Level 3 Eq. 6	Level 2 Eq. 4	Level 1 Eq. 1
1	0:00	0	Zero starting value	70.1	See plot	Zero starting value	Zero starting value
2	1:00	1	= B1	70.0	See plot	= (AMRF-1) / LN (AMRF) * E2 * C2 + AMRF * E1	= A * 43560 * (1-SF) / 3600 * (RD + AVG (F1:F2)) * C2 / 12 + SF * G1
3	2:00	1	=B2	69.9	See plot	= (AMRF-1) / LN (AMRF) * E3 * C3 + AMRF * F2	= A * 43560 * (1-SF) / 3600 * (RD + AVG (F2:F3)) * C3 / 12 + SF * G2
4	3:00	1	=B3	69.8	See plot	= (AMRF-1) / LN (AMRF) * E4 * C4 + AMRF * F3	= A * 43560 * (1-SF) / 3600 * (RD + AVG (F3:F4)) * C4 / 12 + SF * F3

4.2 Long-Term Model Example

This example illustrates the performance of the model for varying antecedent moisture conditions over several months. This shows how the model predicts the rainfall capture fraction changing significantly over time from seasonal effects and back-to-back precipitation events. The computations for this model are too large to tabulate in this paper so graphical output from the model is shown. The full computations can be accessed in Tab 20 of the companion spreadsheet.

For this example, a real rainfall and air temperature time series for an entire year was entered into the model. The time period from spring to early summer is shown in the plots to illustrate the period when antecedent moisture conditions change rapidly. The model was computed for the following input parameters:

Catchment Area (*A*)	=	4000 ac
Hydrograph Half Life (*HHL*)	=	22.76 h
Dry-weather Capture Fraction (*RD*)	=	0.01 unitless
Antecedent Moisture Half Life (*AMHL*)	=	48 h

	Temp [°F]	SHCF [1/in.]
Cold	30	0.05
Hot	70	0.01

The shape factor SF and antecedent moisture retention factor AMRF are computed from Equations 2, and 6, respectively, as shown below. Note that the model timestep is Δt = 1 h.

SF = 0.5 ^ (1/22.76) = 0.970

AMRF = 0.5 ^ (1/48) = 0.986

The Precipitation Averaging Time *PAT* was set to 1 hour, resulting in two precipitation timesteps being averaged and a time to peak *TP* of 2 hours. The Moving Average Temperature function had a Temperature Averaging Time *TAT* of 241 hours, which results in averaging about ten days of temperature. This is important so that short-term temperature fluctuations that can occur during thunderstorms don't impact the long-term seasonal variation in antecedent moisture conditions in the model.

The results of the model output for these inputs and assumptions are shown in Figure 8. The figure shows a stacked graph of the following time series (in order from top to bottom):

- Precipitation input
- Air temperature input
- Computed moving average precipitation
- Computed moving average temperature
- Seasonal hydrologic condition factor output
- Additional capture fraction output
- Flow output

Note in the plot that the air temperature and the Moving Average Temperature are generally increasing during the period shown, and this drives a generally decreasing *SHCF*. This in turn drives a generally decreasing *RW* from seasonal effects, but this varies from event to event. Notice the sequence of back-to-back storms just after March 16, which drive the Additional Capture Fraction *RW* back up, as each subsequent storm increases the wetness conditions. The storm with the largest intensity in the period, occurring just after May 30 has one of the lowest *RW* values, and this drives one the lowest peak flow responses, despite this being the highest intensity storm. This shows how the model adjusts *RW* down to reflect dry antecedent moisture conditions. These dynamics in this model reflect the actual conditions that are observed in rainfall-runoff systems.

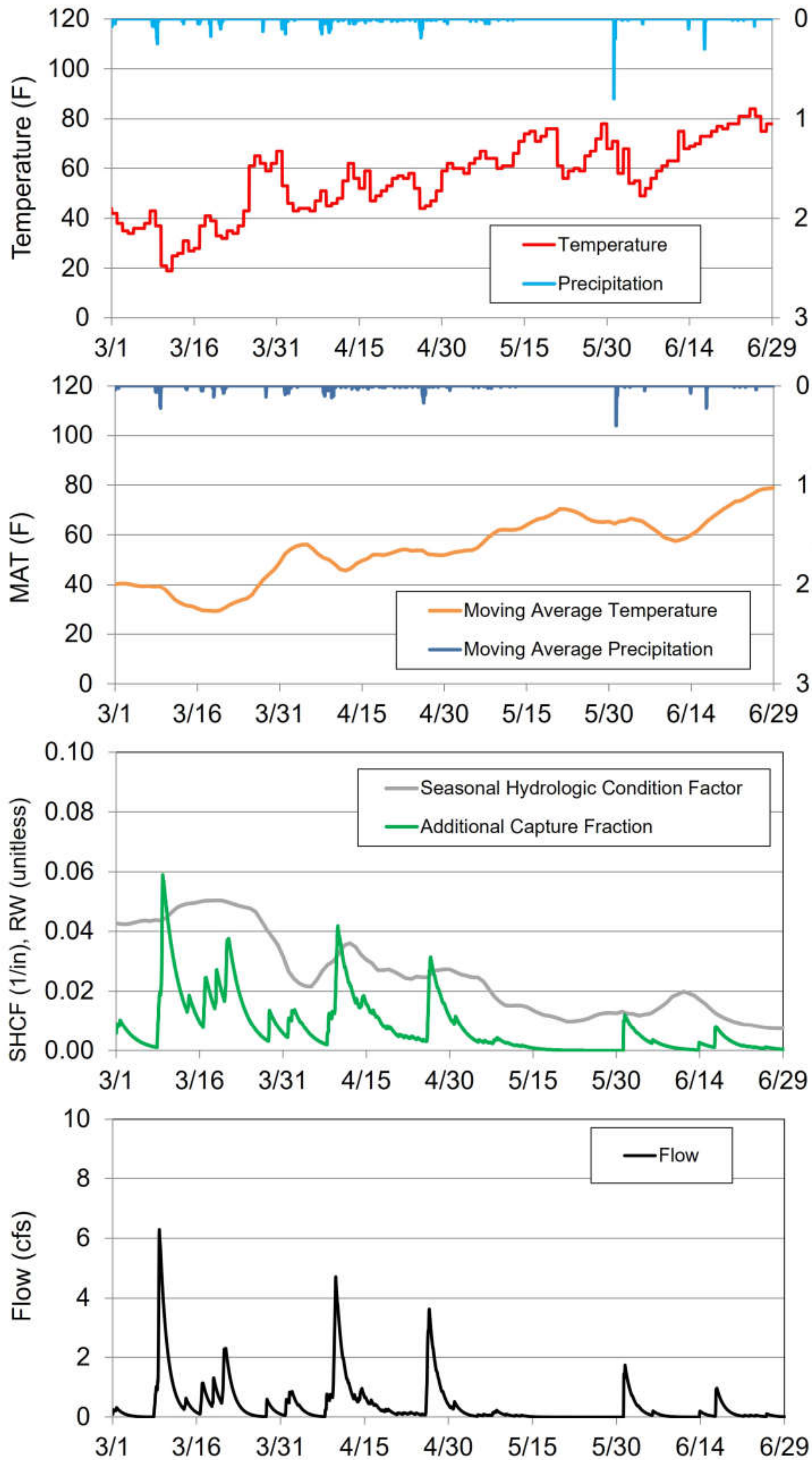
Temperature (F)
Precipitation (inches)
Temperature
Precipitation
MAT (F)
MAP (inches)
Moving Average Temperature
Moving Average Precipitation
SHCF (1/in), RW (unitless)
Seasonal Hydrologic Condition Factor
Additional Capture Fraction
Flow (cfs)
Flow
3/1
3/16
3/31
4/15
4/30
5/15
5/30
6/14
6/29

Figure 8 Plot showing the model inputs and outputs for long-term example.

5 Summary and Conclusions

The Antecedent Moisture Model (AMM) is a relatively simple model that can accurately simulate the impacts of antecedent moisture conditions on the rainfall–runoff process. The reparametrized model equations make the AMM more physically interpretable. For example:

- The catchment area and minimum rainfall capture fraction can be determined directly from measurements made from the system.
- The model can be scaled by changing these parameters, which can be used to simulate changes to the catchment such as sewer separation or sewer rehabilitation.
- The model predicts the continuous variation in the rainfall capture fraction, which is a very common metric used to evaluate collection systems.
- The shape of the hydrograph is described in terms of a half-life rather than a decay, which provides a more intuitive understanding of the system and can be estimated from observed storm hydrographs.
- A precipitation averaging time *PAT* parameter has been added to model a delay between precipitation and the corresponding flow response. The *PAT* plus the time step duration equals the time to peak. *PAT* can be estimated from observed storm hyetographs/hydrographs.

These physical insights provide useful information that can be used by engineers when making design decisions and recommendations. Other changes have also been made to the reparametrized equations to improve usability, including creating timestep independence and updating nomenclature.

6 Future Work

Many opportunities remain to improve the AMM model and its use. Future research is needed in the following areas:

- Comparing AMM results with other hydrologic methods
- Comparing AMM results of different model structures with different configurations of components and number of parameters
- Using streamflow or wastewater treatment plant flows as a seasonal indicator rather than temperature
- Best practices to calibrate sites with short data periods
- Best practices to integrate AMM with design methods
- Estimating AMM parameters for catchments with no flow meter data from physical characteristics of the catchment such as slope, soils, and land cover.
- Performance of AMM when extrapolating to extreme events

7 Resources

The authors have formed an AMM users group for questions regarding using the AMM model and to facilitate continued collaboration and improvement of the model. Interested readers are encouraged to join:

- Users Group: https://groups.google.com/u/0/g/amm-users/

A learning library has been assembled that contains a wealth of information about the model, including a companion spreadsheet, documentation of the equations, and information and videos on how to use the model:

- Learning Library: http://FlowPrediction.com
- Direct link to the companion spreadsheet: https://www.h2ometrics.com/wp-content/uploads/AMM-Master-Companion-Spreadsheet.xlsx

Finally, a Python script implementation of AMM for PCSWMM has been open-sourced. It allows convenient integration of AMM into a PCSWMM modeling workflow and is a useful reference for implementation of the equations into code:

- AMM-for-PCSWMM script: https://github.com/RJNGroup/AMM-for-PCSWMM

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